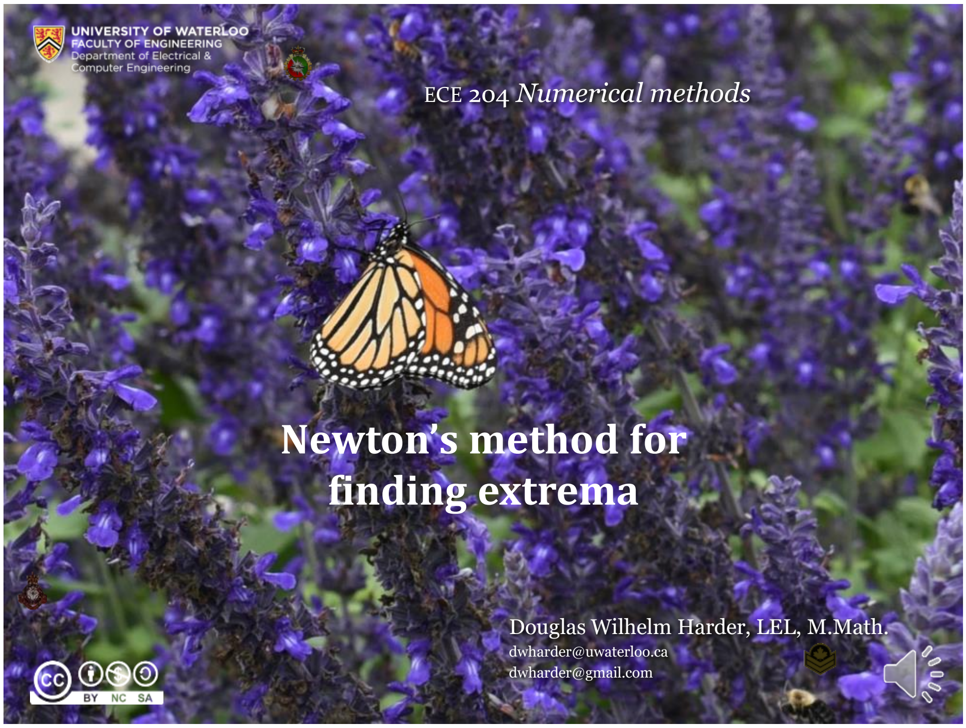


1



# Introduction

- In this topic, we will
  - Describe using Newton's method to find extrema
  - Describe some issues
  - Look at an example



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## Newton's method

- Here is a straight-forward idea:
  - Apply Newton's method to the derivative
- Issue: You're not sure if the point is a maximum, a minimum, or a saddle point
  - A saddle point is where the derivative is zero,
    but the function does not achieve an extremum







## Newton's method

- Given an initial approximation of the extremum $x_0$, let

$$x_{k+1} \leftarrow x_k - \frac{f^{(1)}(x_k)}{f^{(2)}(x_k)}$$

  - If the second derivative is non-zero,
    if it converges, it will converge to an extremum
  - If $f^{(2)}(x_k)$ is the second derivative evaluated at our approximation of the extremum, then
    - It is a minimum if $f^{(2)}(x_k) > 0$
    - It is a maximum if $f^{(2)}(x_k) < 0$
  - If it is a saddle point, we will have O($h$) convergence anyway…

# Issues

- In theory, if $n > 0$ and every derivative up to $f^{(2n)}(x_{\text{ext}})$ is zero, but $f^{(2n)}(x_{\text{ext}}) \neq 0$, it will be an extremum
  - These situations do not usually occur in engineering
- This algorithm will work well if we have formulas for the first and second derivatives
  - This is an excellent place to use automatic differentiation
- If the derivatives must be approximated,
  this technique will amplify that error
  - Other techniques are recommended

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# Error analysis

- Consider this: $f(x_k) = f(x_{\text{ext}}) + \frac{1}{2} f^{(2)}(\xi)(x_k - x_{\text{ext}})^2$

$$f(x_k) - f(x_{\text{ext}}) = \frac{1}{2} f^{(2)}(\xi_1)(x_k - x_{\text{ext}})^2$$

$$f(x_{k+1}) - f(x_{\text{ext}}) = \frac{1}{2} f^{(2)}(\xi_2)(x_{k+1} - x_{\text{ext}})^2$$

$$x_{k+1} - x_{\text{ext}} = -\frac{1}{2} \frac{f^{(3)}(\xi_3)}{f^{(2)}(x_{\text{ext}})}(x_k - x_{\text{ext}})^2$$

$$f(x_{k+1}) - f(x_{\text{ext}}) = \frac{1}{2} f^{(2)}(\xi_2)\left(-\frac{1}{2} \frac{f^{(3)}(\xi_3)}{f^{(2)}(x_{\text{ext}})}(x_k - x_{\text{ext}})^2\right)^2$$

$$= \frac{1}{8} \frac{f^{(2)}(\xi_2)\left(f^{(3)}(\xi_3)\right)^2}{\left(f^{(2)}(x_{\text{ext}})\right)^2}(x_k - x_{\text{ext}})^4$$

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Newton's method for finding extrema

# Example

- Minimize $e^{-x}\cos(x)$ with $x_0 = 1$

| $k$ | $x_k$ | $f(x_k)$ |
|---|---|---|
| 0 | 1 | 0.19876611034641294 |
| 1 | 1.821046307967165 | –0.04008309405526042 |
| 2 | 2.193242198945474 | –0.06503894611173578 |
| 3 | 2.334438237434764 | –0.06698755433628948 |
| 4 | 2.355734523015622 | –0.06701972552457167 |
| 5 | 2.356194278752220 | –0.06701973970827037 |
| 6 | 2.356194490192300 | –0.06701973970827337 |
| 7 | 2.356194490192345 | –0.06701973970827337 |
| 8 | 2.356194490192345 | –0.06701973970827337 |

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Newton's method for finding extrema

# Summary

- Following this topic, you now
  - Are aware Newton's method can be used to find extrema
  - Are aware of the benefits and issues
  - Understand that other techniques are preferable if the derivative and second derivative cannot be precisely
  - Have seen an example

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## References

[1] https://en.wikipedia.org/wiki/Newton%27s_method_in_optimization







## Acknowledgments

None so far.

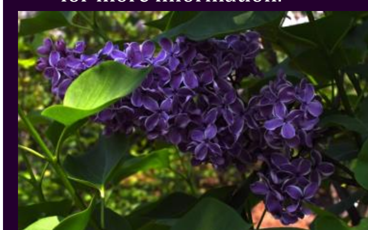
# Colophon

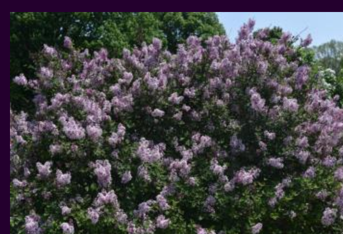
These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.











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